Abelian Chern-Simons theory, Stokes' theorem, and generalized connections[☆]

Hanno Sahlmann[a,1], Thomas Thiemann[b]

[a]*Asia Pacific Center for Theoretical Physics, Pohang (Korea)*
Department of Physics, Pohang University of Science and Technology, Pohang (Korea)

[b]*Institute for Theoretical Physics III, Erlangen University (Germany)*
Max Planck Institute for Gravitational Physics, Potsdam (Germany)
Perimeter Institute for Theoretical Physics, Waterloo (Canada)

Abstract

Generalized connections and their calculus have been developed in the context of quantum gravity. Here we apply them to abelian Chern-Simons theory. We derive the expectation values of holonomies in U(1) Chern-Simons theory using Stokes' theorem, flux operators and generalized connections. A framing of the holonomy loops arises in our construction, and we show how, by choosing natural framings, the resulting expectation values nevertheless define a functional over gauge invariant cylindrical functions.

The abelian theory considered in the present article is the test case for our method. It can also be applied to the non-abelian theory. Results will be reported in a companion article.

Keywords: Abelian Chern-Simons theory, loop quantum gravity, generalized connections

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1. Introduction

One of the pillars of loop quantum gravity is a sophisticated theory of functions on spaces of connections with compact gauge groups. It comprises measure theory [1], the definition of functional derivatives [2], and *spin networks*, generalizations of Wilson loop functionals. Of particular importance to loop quantum gravity is the definition of a (Lebesgue-like) gauge invariant measure on such spaces. It is a natural question whether this formalism can also be applied to other gauge theories. This has been answered affirmatively in a number of cases, such as 2d Yang-Mills theory, or Maxwell theory in four dimensions. With the present article we want to add $U(1)$ Chern-Simons theory on $\mathbb{R}^3$ to this list. In particular we are interested in two (related) questions: (i) can one make sense of the path integral for this theory as a mathematical object related to generalized connections? and (ii) can the formalism be used to *derive* expectation values for the path integral? The point of the present article is to affirmatively answer both questions.

We should point out that $U(1)$ quantum Chern-Simons theory (see for example [3]) in and of itself is not too interesting. We study the abelian theory as a test case for the underlying methods. The application we have in mind is Chern-Simons theory for a non-abelian gauge group, which is more relevant, both from a mathematical and from a physical perspective. In fact, there is already very interesting work on the relation between non-abelian Chern-Simons theory and loop quantum gravity: see for example [4, 5, 6]. We report our own results in this direction in the companion article [15].

$U(1)$ Chern-Simons theory on $\mathbb{R}^3$ is defined by the action

$$S_{\text{CS}}[A] = \frac{k}{4\pi} \int_{\mathbb{R}^3} A \wedge \mathrm{d}A$$

where A is a $U(1)$ connection. The expectation values for a collection of holonomies h_{α_i} along non-intersecting loops $\alpha_1, \ldots, \alpha_n$ can be calculated in various ways to be

$$\langle \prod_i (h_{\alpha_i})^{n_i} \rangle = \exp\left[-\frac{\pi i}{k} \left(\sum_j n_j^2 \operatorname{Link}(\alpha_j, \alpha_j) + 2 \sum_{j<l} n_j n_l \operatorname{Link}(\alpha_j, \alpha_l) \right) \right].$$
(1)

Here Link denotes the Gauß linking number. One nontrivial aspect of the above result is the appearance of the *self-linking* $\operatorname{Link}(\alpha, \alpha)$ of loops which

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is defined as Gauß linking of α with a slightly displaced loop α'. The displaced loop is defined using a *framing* of α. This is the expression, in the U(1) case, of the well known fact that a choice of framing is necessary to compute expectation values of holonomies in Chern-Simons theory. It seems to present an obstacle to using the mathematical methods cited above, in which only unframed loops are considered. One way around it may be to use and extend the formalism of framed spin-networks [7] which provides for framed loops.

Here we will take another route. We will give a derivation of the expectation values (1), using techniques related to generalized connections. The idea is to use Stokes' theorem to replace holonomies under the path integral by (exponentials of) integrals of the curvature over surfaces that have the holonomy loops as boundaries. Then we use the property of Chern-Simons theory that the functional derivative of the action with respect to the connection yields the curvature of the connection. We can thus replace the above-mentioned curvature integrals under the path integral by functional derivatives. Finally we observe that these functional derivatives are well defined objects, known in the loop quantum gravity literature as *flux operators* [8] and can be evaluated, to yield the desired expectation values. On the technical level, it is a connection between Gauß linking and intersection numbers observed quite some time ago [9] that leads to the appearance of the linking numbers in the result. As one can see a prominent role in this derivation is played by surfaces that have a given loop as a boundary. Such surfaces exist for any given loop, but they are not unique. We will demonstrate that the choice of such a surface precisely amounts to the choice of a framing. What is more, we find that there are rules for assignments of surfaces to loops such that the induced framing *only* depends on the topological properties of the loops. Formulated differently, there are ways to assign a framing to a loop which depends just on its topological properties. Such *natural* framings can be incorporated into the definition of the path integral, which, in turn becomes well defined on unframed loops. In this way, we define the Chern-Simons path integral as a path integral[2] on generalized connections.

The structure of the article is as follows: In the next section we will introduce some background material and explain our strategy. In section 3 we carry out the calculation of the expectation values. We describe two examples

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[2]On the technical level, what we obtain is a well defined functional on gauge invariant cylindrical functions.

of natural framings in section 4, and close with a short discussion of our results in section 5. An appendix collects some definitions and technical results related to the surfaces that we use.

2. Strategy

In the present section we will describe our strategy to define the CS path integral. Let us fix the manifold to be $M = \mathbb{R}^3$ and start by listing our ingredients.

(a) We denote by $\overline{\mathcal{A}}$ the space of generalized U(1) connections (see for example [1]). This is a space of distributional connections. It is compact, Hausdorff, smooth connections are dense, and it is well suited for measure theory.

(b) We denote by μ_{AL} the Ashtekar-Lewandowski measure [1], a nondegenerate, uniform measure on the space $\overline{\mathcal{A}}$.

(c) A rigorous definition of the functional derivative $\delta/\delta A$ has been given [8]. More precisely,

$$\widehat{X}_S = \int_S \epsilon_{cab} \frac{\delta}{\delta A_c} \mathrm{d}x^a \wedge \mathrm{d}x^b$$

for any surface S gives a well defined derivation on suitably differentiable functionals of $\overline{\mathcal{A}}$. We will describe its action in more detail below. Here we only need to mention that it has the expected adjointness properties with respect to the scalar product induced by μ_{AL}:

$$\widehat{X}_S^\dagger = -\widehat{X}_S.$$

(d) Stokes' theorem relates the contour integral of an abelian connection A around a loop α bounding a surface S (as in figure 1) to the integral of the curvature over the surface S:[3]

$$\oint_\alpha A = \iint_S \mathrm{d}A.$$

It holds for a very general class of surfaces (so called *domains of integration*, see for example [10]), and in particular for the types of surfaces considered in the present work.

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[3]Since we work on $\mathbb{R}^3$ we can assume that the bundle is trivial and work with the connection as a one-form on the base manifold.
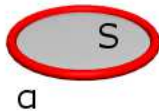
Figure 1: A simple Wilson loop α and a surface S bounded by it

Our strategy is now as follows. To evaluate the expectation value of a Wilson line, we rewrite it as a surface integral over the curvature associated to the connection. Then we observe that

$$\frac{\delta S_{\text{CS}}}{\delta A_c} = \frac{k}{4\pi} F_{ab} \epsilon^{abc}$$

which we use to replace the curvature under the CS path integral with functional derivatives. Finally we apply integration by parts to these derivatives. These manipulations are well motivated but formal. We will see however that they lead to an expression that is well defined and can be evaluated in order to obtain the desired expectation value. Let us be more explicit and consider the CS expectation value of the product of a holonomy

$$h_\alpha[A] \doteq \exp\left[\oint_\alpha A\right]$$

with another functional $F[A]$ of the connection. We stress that the loop α can be arbitrarily knotted.[4] The formal manipulations just described work

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[4]In fact, throughout the text we could also use the word *knot* in place of *loop*, since all the loops are piecewise analytic and hence equivalent to polygonal loops.

out as follows:

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\begin{aligned}
\langle h_\alpha \rangle &= \int_{\overline{\mathcal{A}}} F[A] h_\alpha[A] \exp(i S_{\mathrm{CS}}[A]) \, \mathrm{d}\mu_{\mathrm{AL}}[A] \\
&= \int_{\overline{\mathcal{A}}} F[A] \exp\left[\iint_S F \right] \exp(i S_{\mathrm{CS}}[A]) \, \mathrm{d}\mu_{\mathrm{AL}}[A] \\
&= \int_{\overline{\mathcal{A}}} F[A] \exp\left[\frac{4\pi}{k} \iint_S \frac{\overrightarrow{\delta}}{\delta A} \right] \exp(i S_{\mathrm{CS}}[A]) \, \mathrm{d}\mu_{\mathrm{AL}}[A] \\
&= \int_{\overline{\mathcal{A}}} F[A] \exp\left[-\frac{4\pi}{k} \iint_S \frac{\overleftarrow{\delta}}{\delta A} \right] \exp(i S_{\mathrm{CS}}[A]) \, \mathrm{d}\mu_{\mathrm{AL}}[A] \\
&= \langle \exp\left(\frac{4\pi}{k} \widehat{X}_S \right) F[A] \rangle
\end{aligned}
$$

where S is a surface bounded by α (as in figure 1). In the last line we have expressed our result as the expectation value of a functional differential operator acting on the functional F. As we have indicated above, this operator is a well defined derivation on cylindrical functions. Therefore, if $F[A]$ is, for example, a product of holonomies, it can be easily evaluated. In this way, we will be able to recursively calculate the desired expectation values.

Let us make an important remark regarding the strategy sketched above. It addresses a question that the reader may have had while reading our description of the surface S bounded by the loop α: Could it not happen that α does not bound *any* surface? What then? Indeed, by definition, if α is a non-trivial cycle of the manifold M, it is not the boundary of a surface. This is why we restrict ourselves to $\mathbb{R}^3$ for the present paper. What our approach can say in the case of not simply connected manifolds is an interesting question that may be investigated elsewhere.

Still, even for the case of $M = \mathbb{R}^3$ it may not be immediately obvious that for any loop, one can find a surface bounded by it. This however is assured: there are always such surfaces. One type of surface having a given loop (or even link) as its boundary is called *Seifert surface* of the knot or link. By definition a Seifert surface is embedded, connected, and orientable, the latter of which is important for our purposes since it ensures that the derivation X is well defined. A theorem by Pontrjagin and Frankl asserts that there exists a Seifert surface for any knot or link. (An elegant construction of such a surface is due to Seifert, hence the name.) Two examples of Seifert surfaces are depicted in figure 2. Besides Seifert surfaces, we will also use the notion of a *compressing disc* for a given loop. A compressing disc for

Figure 2: Examples for Seifert surfaces of knots and links: For the trefoil knot (right), and the Hopf rings (left) (graphics created with `SeifertView` [12])

a loop is a surface bounded by the loop, which is an immersion of a disc. More detailed descriptions of the surfaces used in the following, as well as some simple technical results, are compiled in the appendix.

Now that we have ensured that to a given knot or link there are surfaces bounding it, a reasonable question is: Aren't there too many? We will see that the non-uniqueness in assigning a surface to a loop shows up in the end result as a choice of framing for the loop. A surface bounded by the loop induces a framing on the loop[5], and it is this framing that determines the self-linking in (1).

With these remarks in place, we can now move towards the actual calculation.

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[5]Pick any vector field on the loop that is everywhere non-zero, and transversal to the surface. Any such choice will lead to a framing, and all the framings obtained this way are equivalent.

3. Calculation of expectation values

In this section we will obtain the expectation value (1) using language and some techniques from loop quantum gravity. For some technical background on the surfaces used we refer to the appendix. We will work with the action

$$S_{\text{CS}}[A] = \frac{k}{4\pi} \int_{\mathbb{R}^3} A \wedge dA = \frac{k}{8\pi} \int_{\mathbb{R}^3} \epsilon^{abc} A_a \partial_b A_c \, d^3 x$$

and A is a real field. Then

$$\frac{\delta S_{\text{CS}}}{\delta A_c} = \frac{k}{4\pi} F_{ab} \epsilon^{abc}$$

with F_{ab} the components of curvature $F = dA = \partial_a A_b dx^a \wedge dx^b$, hence

$$\epsilon_{cab} \frac{\delta S_{\text{CS}}}{\delta A_c} dx^a \wedge dx^b = \frac{k}{2\pi} F. \tag{2}$$

The integrated functional derivative on the left hand side is a well known object in loop quantum gravity. It can be applied to holonomies, or, more general, cylindrical functions and acts as a derivation $\widehat{X}_S$:

$$\widehat{X}_S = \int_S \epsilon_{cab} \frac{\delta}{\delta A_c} dx^a \wedge dx^b.$$

For the group $U(1)$, an integer n and a loop α,

$$\widehat{X}_S h_\alpha^n = in I(S, \alpha) h_\alpha^n \tag{3}$$

where $I(S, \alpha)$ is the signed intersection number between S and α (see appendix Appendix A). We also note that

$$I(S, \alpha) = \text{Link}(\partial S, \alpha).$$

For a proof see the appendix. We rewrite the right hand side of (2) using Stokes' theorem, and obtain (formally):

$$\frac{2\pi}{k} \widehat{X}_S e^{iS_{\text{CS}}} = i e^{iS_{\text{CS}}} \int_{\partial S} A \equiv i A_{\partial S} e^{iS_{\text{CS}}}$$

Thus it is useful to introduce the operator A_α with commutation relations

$$[\widehat{X}_S, A_\alpha] = I(S, \alpha). \tag{4}$$

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Strictly speaking $A_\alpha = -i \ln h_\alpha$ is not well defined since 0 is in the discrete spectrum of h_α. But we will not be concerned by this here, since some of the manipulations under the path integral are formal anyway. The commutator (4) is zero when $\alpha = \partial S$, since $I(S, \alpha)$ as we have defined it is zero in this case. This is the regularization chosen in loop quantum gravity, but one can make other choices, and we will do so, here. A surface S endows its boundary loop ∂S with a framing in a natural way. Just choose a smooth vector field on ∂S that is transversal to ∂S and nowhere tangent to the surface. Different vector fields chosen this way are equivalent as framings, as one can easily see. Pick one of these vector-fields, v, and use it to "transport" the boundary loop outwards: If $(\partial S)(t)$ is some parametrization of the boundary loop, then $(\partial S)_\epsilon(t) := (\partial S)(t) + \epsilon v(t)$. Let us then define

$$[\widehat{X}_S, A_{\partial S}] = \lim_{t \to 0} [\widehat{X}_S, A_{\partial S_\epsilon}]. \tag{5}$$

Let us also modify our definition of the signed intersection number I such that with the definition (5), the commutation relations between the $\widehat{X}$ and the connection can still be written in the form (3). We also note that (5) leads back to the standard regularization (in which the commutator (5) vanishes) for surfaces without self-intersections.

Now we can calculate the the expectation values we are interested in. Let $\alpha_1 \ldots \alpha_N$ be loops, $S_1 \ldots S_N$ surfaces with $\partial S_i = \alpha_i$, and $n_1 \ldots n_N$ integers. Then formally

$$\langle \prod_i h_{\alpha_i}^{n_i} \rangle = \int \mathrm{d}\mu_{\mathrm{AL}}[A] h_{\alpha_N}^{n_N} \ldots h_{\alpha_2}^{n_2} \left[\sum_{l=0}^\infty \frac{1}{l!} (in_1 A_{\alpha_1})^l \right] e^{iS_{\mathrm{CS}}}$$

$$= \int \mathrm{d}\mu_{\mathrm{AL}}[A] h_{\alpha_N}^{n_N} \ldots h_{\alpha_2}^{n_2} \left[1 + \sum_{l=1}^\infty \frac{1}{l!} (in_1 A_{\alpha_1})^{l-1} \frac{2\pi n_1}{k} \widehat{X}_{S_1} \right] e^{iS_{\mathrm{CS}}}$$

$$= \int \mathrm{d}\mu_{\mathrm{AL}}[A] h_{\alpha_N}^{n_N} \ldots h_{\alpha_2}^{n_2} \left[1 - \frac{2\pi n_1}{k} \sum_{l=1}^\infty \frac{1}{l!} (in_1 A_{\alpha_1})^{l-1} \overleftarrow{\widehat{X}_{S_1}}) \right] e^{iS_{\mathrm{CS}}}$$

where in the last step we have used that $\widehat{X}_{S_1}$ is anti-symmetric with respect to the measure μ_{AL}. As a consequence the derivation now acts on something that is allowed to act, and the next steps commute it through the holonomy operators. From (4),(3)

$$-\frac{2\pi n_1}{k} (in_1 A_{\alpha_1})^{l-1} \overleftarrow{\widehat{X}_{S_1}} = -\frac{2\pi n_1}{k} \left[(l-1) in I(S_1, \partial S_1)(in A_{\alpha_1})^{l-2} + \overleftarrow{\widehat{X}_{S_1}}(in A_{\alpha_1})^{l-1} \right].$$

$$\tag{6}$$

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with the result that we have moved the derivation past all the connection terms coming from the same loop α_1. Before moving the derivation further left, we want to repeat the procedure of exchanging an A_{α_1} for a $\widehat{X}_{S_1}$ and commuting it left through all the A_{α_1}'s, until all of the A_{α_1}'s have been eliminated this way. To this end we will use the recursion formula implicit in (6): It says that under the manipulations described

$$(in_1 A_{\alpha_1})^l =: F(l) = (k-1)XF(l-2) + EF(l-1) \qquad (7)$$

where we have introduced the shortcuts

$$X = -\frac{2\pi n_1^2 i}{k} I(S_1, \partial S_1) \qquad E = -\frac{2\pi}{k} \overleftarrow{\widehat{X}}_{S_1}.$$

Let us also make the definition $F(0) = 1$. Note that with the understanding that the operators $\widehat{X}_{S_1}$ always stand to the left, we can use (7) as if E is a number. (7) is not easily solved explicitly, so we will work with its exponential generating function

$$G(w) := \sum_{l=0}^{\infty} \frac{1}{l!} F(l) w^l.$$

We note that what we are really interested in is

$$G(1) = \sum_{l=0}^{\infty} \frac{1}{l!} F(l).$$

From (7) one can derive that $G(w)$ must satisfy a differential equation:

$$G'(w) = XwG(w) + EG(w)$$

The initial condition is $G(0) = F(0) = 1$. We find the solution

$$G(w) = \exp\left[wE + \frac{w^2 X}{2}\right].$$

By evaluating at $w = 1$ we find that under the path integral, and using the manipulations we have described above,

$$\sum_{l=0}^{\infty} \frac{1}{l!} (in_1 A_{\alpha_1})^l = e^{-\frac{2\pi}{k}\overleftarrow{\widehat{X}}_{S_1}} e^{-\frac{\pi n_1^2 i}{k} I(S_1, \partial S_1)}.$$

Inserting this into the calculation of the expectation value, and pulling the $\widehat{X}_{S_1}$ further to the left, we end up with

$$\langle \prod_i h_{\alpha_i}^{n_i} \rangle = \exp\left[-\frac{\pi n_1^2 i}{k} I(S_1, \partial S_1) - \frac{2\pi}{k} \sum_{j=2}^{n} I(S_1, \alpha_j) \right] \int \mathrm{d}\mu_{\mathrm{AL}}[A] h_{\alpha_N}^{n_N} \dots h_{\alpha_2}^{n_2} e^{iS_{\mathrm{CS}}}.$$

We can now repeat this procedure with the other holonomies, and, as a final step replace signed intersection numbers I by Gauß linking (or, in the case of $I(S, \partial S)$, self-linking) and obtain the well known result

$$\langle \prod_i h_{\alpha_i}^{n_i} \rangle = \exp\left[-\frac{\pi i}{k} \left(\sum_j n_j^2 \operatorname{Link}(\alpha_j, \alpha_j) + 2 \sum_{j<l} n_j n_l \operatorname{Link}(\alpha_j, \alpha_l) \right) \right]. \tag{8}$$

Let us summarize the main points of this section:

- CS expectation values can be calculated using some formal manipulations that are however well motivated from the properties of the objects involved.

- Framing and self-linking enter the picture via choice of surfaces that have a given loop as boundary.

Some remarks: (i) The linking as defined above does not take into account the framing when considering two loops that partially, but not completely, overlap. It is an interesting question if this can be changed while maintaining consistency.
(ii) As far as we can see the formal calculation performed above and its result are insensitive to the order in which the connection terms are exchanged for derivations. They are however not insensitive to more drastic changes. For example one could consider using a Seifert surface for the entire link $\alpha_1 \cup \alpha_2 \dots \cup \alpha_n$, instead of separate surfaces for each loop. In that case, we would find the expectation value to be 1 identically. So this part of the procedure has to be regarded as a regularization ambiguity that we have fixed in a certain way.

4. Natural framings

Obviously, the result (8) depends on the surfaces S_i chosen in the process of calculation. These enter through the self-linking for the loops α_i that

they define. The formalism of loop quantum gravity is using loops (or more generally, graphs) that are not framed. So for (8) to serve as a definition of a functional on Cyl, one will have to make a choice of surface, and hence of framing, for each loop. One way to proceed is to leave the framing unspecified and go over to the formalism of framed spin-networks [7]. But there is also another option: One can search for ways in which loops get assigned surfaces (and hence framings) based on their properties. Such a choice would then be part of the definition of the path integral. A minimal requirement on such a choice is that it makes the expectation values (8) invariant under diffeomorphisms, i.e.,

$$\langle \prod_i h_{\alpha_i}^{n_i} \rangle = \langle \prod_i h_{\phi(\alpha_i)}^{n_i} \rangle$$

for any diffeomorphism ϕ of $\mathbb{R}^3$, which can be connected to the identity. A necessary and sufficient condition in terms of a map $\alpha \longmapsto S_\alpha$ from loops to surfaces is thus

$$I(S_\alpha, \alpha) = I(S_{\phi(\alpha)}, \phi(\alpha))$$

for all loops α and all diffeomorphisms ϕ connected to the identity. Such a map then endows each loop with a framing such that

$$\text{Link}(\alpha, \alpha) = \text{Link}(\phi(\alpha), \phi(\alpha)) \tag{9}$$

for all loops α. Let us mention two examples for such assignments of surfaces and hence framings:

1. For a loop α choose S_α to be a minimal compressing disc, i.e. one that minimizes the number of intersections of the loop with the surface. The minimal number of such intersections is an invariant of the loop, the knottedness $K(\alpha)$ [13], and $\text{Link}(\alpha, \alpha) = K(\alpha)$.
2. For a loop α choose S_α to be a Seifert surface. A Seifert surface does not self-intersect and hence $\text{Link}(\alpha, \alpha) = 0$.

We call the framing obtained from these and similar prescriptions *universal* and *natural*, the former because they encompass all loops, the latter because of their covariance (9). (The second prescription also yields what is called *natural framing* in the mathematical literature.) Making one of these choices, (8) becomes a function of unframed loops. Still it is not a functional on Cyl since not all functionals in Cyl are linear combinations of multiloops. It is however well known that the gauge invariant functionals can all be written as such linear combinations and on those (8) defines a well defined functional.

We remark that for graphs that can be decomposed into loops, the above prescriptions will also give a notion of 'framed graph', and hence also a notion of framing for gauge invariant functionals in Cyl. It would be interesting to compare this in detail to the notion of framed spin network of [7]. We suspect that the notions will turn out to be the same.

5. Closing remarks

In the present article we have done two things. On the one hand we have explained how the Wilson loop expectation value (1) of U(1) Chern-Simons theory can be interpreted as a functional over gauge invariant cylindrical functions by using a universal framing prescription, and we have given two examples of such a prescription. On the other hand we have *derived* the expectation values (1) using some heuristic manipulations of the path integral of the theory. These manipulations used the fact that certain functional derivatives have well defined action on Wilson loops in the form of *flux operators*, well known from loop quantum gravity.

Since U(1) Chern-Simons theory is not very interesting in itself, the potential value of the work presented above lies in the methods used. Using a universal framing prescription to describe a theory that needs framing with the mathematical methods based on generalized connections may be useful for different theories. And we have used a regularization of the (abelian) flux that differs from the one used in LQG. This may also find applications elsewhere. But, most importantly, the technique we have used to compute the expectation values (1) seem to be applicable also to Chern-Simons theory with compact, *non*-abelian gauge group. In particular, in [15] we have started to investigate the case of SU(2), relevant to knot theory [14], Euclidean gravity in three dimensions, and maybe even to four dimensional gravity via the Kodama state. Our findings show that again flux operators can be defined that replace holonomies under the path integral, using a non-abelian generalization of Stokes' theorem. The regularization of these operators is however much more complicated than in the abelian case. In particular, since the functional derivatives do not commute for the non-abelian group, there is an ordering ambiguity. In fact, something like it is expected since it may be the technical reason for the occurrence of quantum SU(2) in the quantum theory. In [15], we show that a certain natural mathematical structure in the theory of Lie, algebras – the so-called *Duflo*

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map – can be used to perform the ordering, and we calculate the results for some simple cases.

Acknowledgements

Appendix A. Surfaces, Gauß linking, and intersection numbers

Before we start the calculation of expectation values in CS theory, we will collect here some definitions and mathematical facts needed in the following sections.

For the moment a loop will be a smooth, compact closed one dimensional submanifold in $\mathbb{R}^3$. Also we assume loops to be tame, i.e. equivalent to a polygonal knot. Later we will have to make some adjustments to this definition owing to the fact that loops in the formalism of loop quantum gravity that we are going to use are actually piecewise analytic.

Let us pick an orientation of $\mathbb{R}^3$ and stick with it throughout. All the surfaces we consider will be orientable, and we assume them to be oriented, even if we do not state this explicitly each time. We will also take the loops to be oriented.

Given a loop α in $\mathbb{R}^3$ we consider surfaces S such that $\partial S = \alpha$. We will always assume that S and α are oriented consistently.[6] We will use two classes of such surfaces, Seifert surfaces, and compressing discs. Let us give a brief description of each.

Definition Appendix A.1. *A Seifert surface for a loop α is an orientable, connected submanifold S such that $\partial S = \alpha$.*

Seifert surfaces exist for any loop in $\mathbb{R}^3$. In fact there exist a simple algorithm to construct one for a given loop. There is however no uniqueness, not even

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[6]That means, if T is a positively oriented tangent vector to the loop α at a point p, and N is the outward normal vector (tangent to S) of S at p, then (N, T) is a positively oriented basis of $T_p S$.

in a topological sense: A loop has many different Seifert surfaces. For these and other general results on Seifert surfaces see for example [11].

For the definition of a compressing disc we follow [13]. That reference works with loops in S^3 instead of $\mathbb{R}^3$ but for the few things we need from there, the compactification makes no difference.

Definition Appendix A.2. *We define a compressing disc of a loop α to be a map f from the two dimensional disc D^2 into $\mathbb{R}^3$ such that $f|_{\partial D^2} = \alpha$ and $f|_{intD^2}$ is transverse to α.*

Then $f(\text{int} D^2)$ has only finitely many intersections with α, and one can show that for a given loop there is a minimal number of intersections that can be achieved by varying the compressing disc. This number is called knottedness and is an invariant of the loop. It is shown in [13] that starting from a compressing disc with the image of which has n intersections with the loop, one can always find a compressing disc with n or less intersections, which in addition is an immersion of D^2 into $\mathbb{R}^3$. In the following (and in the main text) we will always assume all the compressing discs to be immersions.

Let us call an orientable, connected, two-dimensional submanifold of $\mathbb{R}^3$ a surface of *type I*, and an immersion of D^2 into $\mathbb{R}^3$ a surface of *type II*. A notion we need for both types of surfaces is that of the *signed intersection number* between a surface and a loop.

Definition Appendix A.3. *For a surface S of type I and a loop α, the signed intersection number is*

$$I(S, \alpha) = \sum_{p \in S \cap \alpha} \kappa(p)$$

where $\kappa(p) = +1$ if the intersection is transversal and the orientations of α and S together give the orientation of the one chosen on $\mathbb{R}^3$,[7] $\kappa(p) = -1$ if the intersection is transversal and the orientations of α and S together give the orientation opposite of the one $\mathbb{R}^3$, and zero otherwise.

For a surface of type II the definition is almost the same, except that intersections with a loop at a point of self-intersection of the surface may count multiple times.

Definition Appendix A.4. *Let $f : D^2 \to \mathbb{R}^3$ be a surface of type II. Then we can find an open cover $\{U_J\}$ of D^2 such that $f(U_J)$ is a surface of type*

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[7]We mean: If T is a positively oriented tangent vector to α at the intersection point p, and (v_1, v_2) is a positively oriented basis of $T_p S$, then (T, v_1, v_2) is positively oriented in $T_p\mathbb{R}^3$.
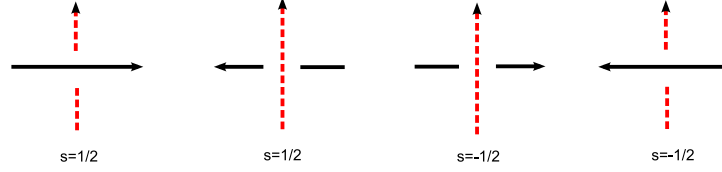
Figure A.3: The types of crossings used in the calculation of the linking number of two loops.
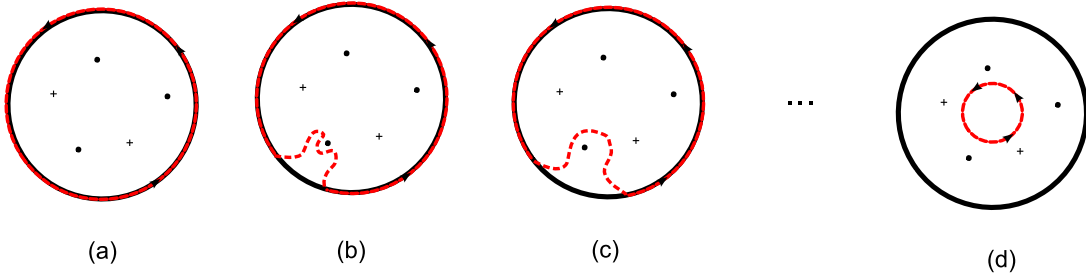

Figure A.4: The shrinking procedure of the loop used in the proof of prop. Appendix A.5. Orientation of D^2 is out of this page, towards the reader, intersections with positive signature are indicated by a '+', those with negative signature by a dot.

I for every J. Then given a loop α we define

$$I(S, \alpha) = \sum_J I(f(U_J), \alpha).$$

It is easily checked that this definition is independent of the open cover.

Another notion that we will need is that of the Gauß-linking number (or simply linking number). It is a property of a pair of oriented loops. One of several equivalent ways of defining it is the following: Given two oriented loops α_1, α_2, draw a diagram of the pair of loops. To each crossing c in the diagram, determine the quantity s_c by comparison with figure A.3. Then the linking number is

$$\text{Link}(\alpha_1, \alpha_2) = \sum_c s_c.$$

It is easy to see that the linking number is independent of the diagram chosen, and that it is, in fact, a topological invariant of the loops.

The linking number is related to the signed intersection number as follows:

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Proposition Appendix A.5. *Let S be a surface of type I or II, and α be a loop. Then*

$$I(S, \alpha) = \text{Link}(\partial S, \alpha).$$

Proof. For a surface of type I, the proof of the proposition is contained in the paper [9] as appendix 1, and so we won't reproduce it here. For surfaces of type II the argument is as follows: Let a type II surface S and a loop α be given. Thus we have an immersion f of D^2 into $\mathbb{R}^3$, the image being S. Let us describe the situation by depicting D^2, together with the preimage of the intersection points of S with α. Thus we get a picture like figure A.4(a), where we have additionally kept track of the signature κ of the intersection. The signed intersection number can be read of from the diagram, by subtracting the number of dots (preimages of negative signature intersections) from the number of pluses (preimages of positive signature intersections). Now we will construct a family of new type II surfaces $S(t)$ from S, by suitably restricting the domain of the immersion f, or, in other words by shrinking the boundary loop. To this end, we consider a homotopy of loops $\beta(t)$ in D^2, with $\beta(0) = \partial D^2$ and $\beta(1)$ a loop that has no preimages of intersections within the disc that it bounds (such a loop is depicted in A.4(d)), and such that its image under f has no linking with α. Obviously the latter two conditions are compatible, and many such loops exist. We define the surface $S(t)$ as the image under f of the disc bounded by $\beta(t)$. Since f is only an immersion, $f(\beta(t))$ for any given $t \neq 0$ is not necessarily a loop: It can have self-intersections. One can convince oneself, however, that one can choose the homotopy in such a way that the self-intersections of $f(\beta(t))$, if present at all, are isolated points, and such that $f(\beta(1))$ does not have any self-intersections. We will consider such a homotopy in the following. Let us keep track of $\text{Link}(\alpha, \beta(t))$ and $I(S(t), \alpha)$ as we vary t Obviously we start with

$$\text{Link}(\alpha, f(\beta(0))) = \text{Link}(\alpha, \partial S) \qquad I(S(0), \alpha) = I(S, \alpha).$$

As long as $f(\beta(t))$ does not develop self-intersections or $\beta(t)$ moves past the pre-image of an intersection of S with α, nothing changes. When $f(\beta(t))$ develops a self-intersection, $\text{Link}(\alpha, \beta(t))$ is a priori not well defined. But we will define it to be whatever one gets when one removes the self-intersections by slightly moving $f(\beta(t))$ such that the self-intersections disappear, without however moving $f(\beta(t))$ through α. The result is independent of the precise way this done, since Link can be computed using only the crossings of α with $f(\beta(t))$, no self-crossings of $f(\beta(t))$. $I(S(t), \alpha)$ is obviously insensitive

to self-intersections of $f(\beta(t))$. If on the other hand $\beta(t)$ moves past the pre-image of an intersection of S with α (as depicted in figures A.4(b),(c)), both quantities change: For a positive signature intersection,

$$I(S(t_{\text{after}}), \alpha) = I(S(t_{\text{before}}), \alpha) - 1$$

and, by studying figure A.3,

$$\text{Link}(f(\beta(t_{\text{after}})), \alpha) = \text{Link}(f(\beta(t_{\text{before}})), \alpha) - 1.$$

Similarly, for passing a negative signature intersection,

$$I(S(t_{\text{after}}), \alpha) = I(S(t_{\text{before}}), \alpha) + 1$$
$$\text{Link}(f(\beta(t_{\text{after}})), \alpha) = \text{Link}(f(\beta(t_{\text{before}})), \alpha) + 1.$$

At the end of the process,

$$\text{Link}(\alpha, f(\beta(1))) = 0, \qquad I(S(1), \alpha) = 0.$$

Thus $\text{Link}(\alpha, \beta(t))$ and $I(S(t), \alpha)$ are the same in the end, and they change in step, thus they have been equal at the beginning as well, which proves the proposition. $\qquad\square$

A final remark of this appendix concerns the differentiability category used for the loops and surfaces. In the description above all surfaces (and implicitly also their boundaries) are smooth. In loop quantum gravity the surfaces for which the flux operators are well defined are however real analytic (or piecewise real analytic, defined in a suitable sense), and so are the loops. This is to ensure that there are only finitely many transversal intersections for all pairs of compact loops and surfaces, which is in turn necessary to make the analog, for non-abelian groups, of the commutator (3) well defined. This is however not a concern for the work presented here, since due to the abelian nature of U(1), flux operators are well defined even on loops that intersect the underlying surface infinitely often as long as the Gauss linking between the loop and the boundary of the surface is finite. This is the case if we work with piecewise analytic loops and boundaries. Thus the remaining question is whether the existence of Seifert surfaces and compressing discs, as well as the results of this appendix continue to hold for piecewise analytic loops. It is easy to see that Seifert surfaces continue to exist in this case, and the notion of a compressing disc can be trivially generalized to allow for a piecewise analytic boundary. Finally it is easy to see that all properties remain intact. Thus, in the main text, we will always assume loops to be piecewise analytic and surfaces to be smooth.

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